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ON THE LEAST PRIME IN AN ARITHMETICAL PROGRESSION

By S. Chowla.

Let k and l be integers prime to each other, of which k is positive. Dirichlet proved that there are infinitely many primes in the arithmetical progression $kx+l$. If $P(k, l)$ is the least prime of this form it is probable that $P(k, l) < k^{1+\epsilon}$ for every positive ϵ and all large k. We are very far at present even from being able to show that $P(k, l) < k^m$ where m is any fixed positive constant independent of k, which is large. However if we assume the truth of the so-called "extended Riemann hypothesis" we can(1) show that $P(k, l) < k^{2+\epsilon}$ for $k > k_0(\epsilon)$.

Taking the special case $l=1$ we can show that $P(k, 1) < e^{Ak}$ where A is a positive constant independent of k, in the special case when k is a prime as follows. We know that every prime factor of $2^k - 1$ is $\equiv 1 \pmod{k}$ when k is a prime. Hence $P(k, 1) < 2^k$ and the result stated follows. It is improbable that such a simple argument exists for general l.

In what follows I show that

$$P(k, l) < e^{A k^{\frac{3}{2}} \log^6 k} \tag{I}$$

for primes $k \equiv 3 \pmod 4$. It is likely that my proof of (I) can be extended without difficulty to general k. I have not bothered to do this since much more can be proved by an appropriate revision of the proof of the generalized Prime Number Theorem giving the number of primes $\equiv l \pmod k$ not exceeding x. The results of this investigation will appear elsewhere.

§. 1. *Proof of (I):* Throughout what follows k is a prime $\equiv 3 \pmod 4$, $\chi(n)$ is a Dirichlet's character (mod k). χ_0 is the principal character, χ_1 denotes any non-real character, χ_2 denotes (there is only one such when k is a prime) a real non-principal

---

[1] This follows from Theorem 6 of Titchmarsh's paper "A divisor problem", *Rendiconti del Circolo Matematico di Palermo*, Tomo LIV, 1930, 414-429.

character. Numerals in thick print are references to the equations of this paper.

Lemma 1: We have(2)

$$(1) \qquad \sum_{a}^{b} \chi(n) = O(\sqrt{k} \log k).$$

Lemma 2: *We have*(3)

$$(2) \qquad \frac{1}{L_{x_1}(1)} = O(\log^6 k)$$

Lemma 3:

$$(3) \qquad \frac{1}{L_{x_2}(1)} = O(\sqrt{k})$$

Proof: We have

$$L_{\chi_2}(1) = \pi \frac{\sum (b-a)}{k \sqrt{k}} \geqq \frac{\pi}{\sqrt{k}}$$

(where b runs through the quadratic non-residues of k and a through the residues of k) since, for primes k

$$\Sigma b \equiv \Sigma a \equiv O \pmod{k}$$

and $L_{\chi_2}(1) \neq O$.

For $\chi \neq \chi_0$ we have(4)

$$\sum_{n \leq x} \frac{\chi(n) \log n}{n} = \sum_{p^m \leq x} \frac{\chi(p^m) \log p}{p^m} \sum_{n \leq \frac{x}{p^m}} \frac{\chi(n)}{n}$$

$$= \sum_{p^m \leq x} \frac{\chi(p^m) \log p}{p^m} \left(L_x(1) + O\left(\frac{\sqrt{k} \log k\, p^m}{x}\right)\right) (1)$$

$$= L_\chi(1) \sum_{p \leq x} \frac{\chi(p) \log p}{p} + O(\sqrt{k} \log k), \sum_{p^m \leq x} \frac{\log p}{x}$$

$$(4) \quad = L_x(1) \sum_{p \leq x} \frac{\chi(p) \log p}{p} + O(\sqrt{k} \log k),$$

---

[2] Landau, *Vorlesungen uber Zahlentheorie*, Bd. 3, S. 178.

[3] Landau, *Math. Annalen*, 70, 1911, 69—78.

[4] Landau, *Primzahlen*, 447—8.

$$\sum_{\substack{p\equiv l(\mathrm{mod}\ k)\\ p\leqslant x}}\frac{\log p}{p}=\frac{1}{S(k)}\sum_{\chi=1}^{S(k)}\frac{1}{\chi_\chi(l)}\sum_{p\leqslant x}\frac{\chi_\chi(p)\log p}{p}\ ^{(5)}$$

$$=\frac{1}{S(k)}\frac{1}{\chi_0(l)}\sum_{p\leqslant x}\frac{\chi_0(p)\log p}{p}+\frac{1}{k-1}\sum_{\chi=\chi_1}\frac{1}{\chi(l)}\sum_{p\leqslant x}\frac{\chi(p)\log p}{p}$$

$$+\frac{1}{k-1}\frac{1}{\chi_2(l)}\sum_{p\leqslant x}\frac{\chi_2(p)\log p}{p}$$

$$(5)\quad =\frac{\log x+O(1)}{k-1}+O(\sqrt{k}\log^6 k)\quad (1,\ 2,\ 3).$$

(I) follows immediately from (5).

---

[5] $S(k)$ is the number of positive integers not exceeding k and prime to it.

ON A CERTAIN POLYNOMIAL ANALOGOUS TO LOMMEL'S POLYNOMIAL.

By Dr. S. C. Mitra, ph.d.

1. In a previous paper(1) the author has shown that the parabolic cylinder function $D_n(x)$ satisfies the relation

$$D_{n+n}(x)+(-1)^m(n+1)\,R^{m}_{n,\,m-2}(x)\,D_n(x)+(-1)^m R^{(x)}_{n-1,\,m-1}\,D_{n+1}(x)=O,$$

where $R_{n,\,m}(x)$ is a polynomial analogous to that of Lommel in the Theory of Bessel Functions. The differential equation satisfied by $R_{n,\,m}(x)$ has been given in a subsequent paper(2) by Mr. R. S. Varma. The object of the present note is to show that a similar relation holds between the parabolic cylinder functions $D_{-(m+n+1)}(ix)$. $D_{(n+1)}(ix)$ and $D_{-(n+2)}(ix)$ and to obtain an interesting property of these functions.

2. Suppose n is a positive integer.

The functions $D_{-(n+1)}(\pm ix)$ satisfy the recurrence formulæ.

$$D_{-n}(ix)-ix\,D_{-(n+1)}(ix)-(n+1)\,D_{-(n+2)}(ix)=O,$$

and

$$D_{-n}(-ix)+ix\,D_{-(n+1)}(-ix)-(n+1)\,D_{-(n+2)}(-ix)=O. \qquad (1)$$

From these formulae, we get

$$\{D_{-(n+2)}(ix)\,D_{-(n+1)}(-ix)+D_{-(n+2)}(-ix)\,D_{-(n+1)}(ix)\}$$
$$=\frac{1}{n+1}\{D_{-(n+1)}(ix)\,D_{-n}(-ix)+D_{-(n+1)}(-ix)\,D_{-n}(ix)\}$$

---

1 "On the properties of a certain polynomial analogous to Lommel's polynomial," *Indian Physico-Mathematical Journal*, Vol. 3, No. I (1932), pp. 9-15.

2 "On a certain polynomial analogous to Lommel's polynomial", *Journal of the Indian Mathematical Society*, Vol. 19, No. 12, December, (1932), p. 274.

$$= \frac{1}{(n+1)\ n\ (n-1)\ \ldots\ 1} \left\{ D_{-1}(ix) D_0(-ix) + D_{-1}(-ix) D_0(ix) \right\}$$

$$= \frac{\sqrt{2\pi}}{\Gamma(n+2)} \qquad (2)$$

The functions $D_{-(m+n+1)}(\pm ix)$ satisfy the differential equation

$$\frac{d^2y}{dx^2} + (m+n+\tfrac{1}{2} - \tfrac{1}{4}x^2)y = 0. \qquad (3)$$

As a solution of (3), let us assume

$$y = D_{-(n+1)}(ix)\ A(x) + i\, D_{-(n+2)}(ix)\ B(x).$$

Substituting in (3), we get, after simplification, the equations

$$A''(x) + xA'(x) + mA(x) + 2B'(x) = 0,$$

and

$$B''(x) - xB'(x) + (m-1)\ B(x) - 2(n+1)A'(x) = 0. \qquad (4)$$

From these we find that $A(x)$ and $B(x)$ satisfy the differential equations,

$$A^{IV}(x) + (4n+2m+4-x^2)\ A''(x) - 3x\ A'(x) + m(m-2)\ A(x) = 0, \qquad (5)$$

and

$$B^{IV}(x) + (4n+2m+2-x^2)\ B''(x) - 3x\ B'(x) + (m^2-1)\ B(x) = 0. \qquad (6)$$

The equation (5) can be obtained from (6) by writing $m-1$ and $n+1$ for m and n respectively. So we shall solve equation (6) only.

To solve this equation, let us assume

$$B(x) = \Sigma C_r\ H_r(x).$$

We find

$$B(x) = C_0 \left[H_0(x) - \frac{(m-1)\ (m+1)}{2!\ (4n+2m-2)} H_2(x) + \frac{(m-1)\ (m-3)\ (m+1)\ (m+3)}{4!\ (4n+2m-2)\ (4n+2m-6)} H_4(x) - \ldots \right]$$

$$+ C_1 \left[H_1(x) - \frac{(m-2\ (m+2)}{3!\ (4n+2m-4)} H_3(x) + \frac{(m-2)\ (m-4)\ (m+2)\ (m+4)}{5!\ (4n+2m-4)\ (4n+2m-8)} H_5(x) - \ldots \right] \qquad (7)$$

From this we deduce

$$A(x) = A_0 \left[H_0(x) - \frac{m\ (m-2)}{2!\ (4n+2m)} H_2(x) \right.$$

$$+\frac{m(m+2)\ (m-2)\ (m-4)}{4!(4n+2m)\ (4n+2m-4)}H_4\ (x)-\ldots\Big]$$

$$+\ C_1\ \Big[H_1(x)-\frac{(m-2)\ (m+2)}{3!\ (4n+2m-4)}H_3(x)$$

$$+\frac{(m-2)\ (m-4)\ (m+2)\ (m+4)}{5!\ (4n+2m-4)\ (4n+2m-8)}H_5\ (x)-\ldots\Big] \qquad (8)$$

3. Let m be a positive integer. Leaving aside the details of actual calculations, we can easily prove that when m is an odd integer,

$$D_{-(m+n+1)}^{(ix)} = A_1\ \Big[H_1\ (x)-\frac{(m-3)\ (m+1)}{3!\ (4n+2m-2)}H_3\ (x)$$

$$+\frac{(m-3)\ (m-5)\ (m+1)\ (m+3)}{5!\ (4n+2m-2)\ (4n+2m-6)}H_5\ (x)-\ldots\Big]D_{-(n+1)}^{(ix)}$$

$$+iC_0\ \Big[H_0\ (x)-\frac{(m-1)\ (m+1)}{2!\ (4n+2m-2)}H_2(x)$$

$$+\frac{(m-1)\ (m-3)\ (m+1)\ (m+3)}{4!\ (4n+2m-2)\ (4n+2m-6)}H_4\ (x)-\ldots\Big]D_{-(n+2)}^{(ix)},$$

and when m is an even integer,

$$D_{-(m+n+1)}^{(ix)} = A_0\ \Big[H_0(x)-\frac{(m-2)m}{2!\ (4n+2m)}H_2(x)$$

$$+\frac{(m-2)\ (m-4)m(m+2)}{4!(4n+2m)\ (4n+2m-4)}-\ldots\Big]D_{-(n+1)}^{(ix)}$$

$$+iC_1\ \Big[H_1(x)-\frac{(m-2)\ (m+2)}{3!\ (4n+2m-4)}H_3(x)$$

$$+\frac{(m-2)\ (m-4)\ (m+2)\ (m+4)}{5!\ (4n+2m-4)\ (4n+2m-8)}H_5(x)-\ldots\Big]D_{-(n+2)}^{(ix)}, \qquad (A)$$

where

$$A_0=\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\ \frac{(m-2)!\ (4n+2m)\ (4n+2m-4)\ldots(4n+8)}{(m-2)\ (m-4)\ldots 2m\ (m+2)\ldots(2m-4)}$$

$$C_0=-i\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\ \frac{(m-1)!\ (4n+2m-2)\ (4n+2m-6)\ldots(4n+4)}{(m+1)\ (m+3)\ldots(2m-2)\ (m-1)\ (m-3)\ldots 2}$$

$$A_1=-i\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\ \frac{(m-2)!\ (4n+2m-2)\ (4n+2m-6)\ldots(4n+8)}{(m+1)\ (m+3)\ldots(2m-4)\ (m-3)\ (m-5)\ldots 2}$$

$$C_1=-\frac{\Gamma(n+2)}{\Gamma(m+n+1)}\ \frac{(m-1)!\ (4n+2m-4)\ (4n+2m-8)\ldots(4n+4)}{(m-2)\ (m-4)\ldots 2\ (m+2)\ (m+4)\ldots(2m-2)}$$

4. Writing $-x$ for x, we get exactly similar relations involving $D_{-(m+n+1)}^{(-ix)}$, $D_{-(n+1)}^{(-ix)}$ and $D_{-(n+2)}^{(-ix)}$. We now deduce the following interesting relations from (A).

When m is an odd integer,

$$D_{-(m+n+1)}(ix)\, D_{-(n+1)}(-ix) + D_{-(m+n+1)}(-ix)\, D_{-(n+1)}(ix)$$

$$= \frac{\sqrt{2\pi}\, i\, b_0}{\Gamma(n+2)}\left[H_0(x) - \frac{(m-1)(m+1)}{2!\,(4n+2m-2)} H_2(x)\right.$$

$$\left. + \frac{(m-1)(m-3)(m+1)(m+3)}{4!\,(4n+2m-2)(4n+2m-6)} H_4(x) - \ldots\right],$$

$$D_{-(m+n+1)}(ix)\, D(-ix)_{-(n+2)} - D_{-(m+n+1)}(-ix)\, D_{-(n+2)}(ix)$$

$$= \frac{\sqrt{2\pi}}{\Gamma(n+2)} A_1\left[H_1(x) - \frac{(m-3)(m+1)}{3!\,(4n+2m-2)} H_3(x)\right.$$

$$\left. + \frac{(m-3)(m-5)(m+1)(m+3)}{5!\,(4n+2m-2)(4n+2m-6)} H_5(x) - \ldots\right]. \qquad \text{(B).}$$

When m is an even integer,

$$D_{-(m+n+1)}(ix)\, D(-ix)_{-(n+1)} - D_{-(m+n+1)}(-ix)\, D_{-(n+1)}(ix)$$

$$= \frac{\sqrt{2\pi}}{\Gamma(n+2)}\, i\, C_1\left[H_1(x) - \frac{(m-2)(m+2)}{3!\,(4n+2m-4)} H_3(x)\right.$$

$$\left. + \frac{(m-2)(m-4)(m+2)(m+4)}{5!\,(4n+2m-4)(4n+2m-8)} H_5(x) - \ldots\right],$$

$$D_{-(m+n+1)}(ix)\, D(-ix)_{-(n+2)} + D_{-(m+n+1)}(-ix)\, D_{-(n+2)}(ix)$$

$$= \frac{\sqrt{2\pi}}{\Gamma(n+2)} A_0\left[H_0(x) - \frac{(m-2)m}{2!\,(4n+2m)} H_2(x)\right.$$

$$\left. + \frac{(m-2)(m-4)m(m+2)}{4!\,(4n+2m)(4n+2m-4)} H_4(x) - \ldots\right]. \qquad \text{(C).}$$

The results are true for all positive values of n.

REARRANGEMENT OF COMPLEX SERIES

by

V. GANAPATHY IYER

I. This is an attempt at an exhaustive classification of infinite series whose terms are complex numbers, according to the manner of distribution of the limiting values of the series when the terms of the series are re-arranged so as to form another series. By the limiting values of the series are meant the limit points of the sequence obtained by taking the sum to first n terms of the series in any particular rearrangement. A definite arrangement of the terms of the series being presupposed, the series converges or diverges according as the sequence has a unique finite or infinite limit; otherwise the series oscillates. In the case of complex series, infinity is said to be the unique limit if the *modulus* of the sum to first n terms of the series diverges to $+\infty$. No distinction is made between $\pm\infty$ or infinity in any other direction. This paper contains a generalisation of a theorem given by Steinitz.[1] The method of proof is practically the same as that given in the paper quoted. A simple geometric proof of the result in Lemma (v) below is given without introducing general considerations on the properties of convex sets of points.

II. Real series: Let $A=\sum_{i=1}^{\infty} a_i$ be a real series. If it converges and no rearrangement alters the convergence or the sum, the series is said to be unconditionally convergent. It is known that the necessary and sufficient condition for unconditional convergence is absolute convergence *i.e.* $\sum_{1}^{\infty} |\, a_i \,|$, converges. So we may speak of the two as equivalent. If a series converges but not absolutely then it is said to converge non-absolutely or conditionally. Similarly if A diverges to $\pm\infty$ and no rearrangement disturbs this divergence, the series is said to be absolutely

(1) Steinitz, Bedingt konvergente Reihen und konvexe Systeme. *J. fur die reine und angewandte. Mathematik,* B and 143. (pp. 128—175) § 14.

divergent.(2) With these definitions we may arrive at the classification of real series as follows. It is to be noted that nothing is assumed about the convergence of A in any rearrangement:—

Let $A=\sum_1^\infty a_n$ be any real series where $a_n \to 0$ as $n \to \infty$. Let $B=\Sigma b_n$ and $C=\Sigma c_n$ be the series formed from A by grouping together the positive and negative terms together. Here c_n denotes the numerical value of the corresponding negative term in A. Then three cases can arise:

(i) B and C both converge. Then obviously $\Sigma |a_n|$ converges and A is absolutely convergent.

(ii) One of B and C diverges while the other converges. Then the series diverges to $\pm\infty$(3) and no rearrangement can alter this fact. Hence this series is absolutely divergent.

(iii) Both B and C diverge. In this case the series could be rearranged so as to converge to any real sum or diverge or oscillate finitely or infinitely(4). In this case the series converges conditionally. The usual proof given assumes the convergence of the series A in some rearrangement but this is not essential. The proof(4) depends only on the fact that B and C diverge and $a_n \to o$.

III. Complex Series: (1) In the following Greek letters $\alpha, \beta, \gamma \ldots$ will denote complex numbers and $a, b, c \ldots$ real numbers.

Let $a_1, a_2, a_3 \ldots$ be a sequence of complex numbers. By a partial sum A of these numbers is meant the sum of any finite number of these terms picked out arbitrarily. The system of such partial sums will be denoted by Λ. Suppose in any particular arrangement of the sequence, say $a_1, a_2 \ldots, \sum_1^\infty a_n$ converges. If no rearrangement alters the convergence or the sum, the series is said to be absolutely or unconditionally convergent. Otherwise it is said to be non-absolutely or conditionally convergent. It is easily seen by splitting Σa_n into its real and imaginary parts that the necessary and sufficient condition for absolute or unconditional

(2) This conception of absolute divergence, which secures completeness in the statement of results, was suggested by Dr. R. Vaidyanathaswami.

(3) A will diverge to $\pm\infty$ if B diverges and C converges; and to $-\infty$ if the reverse be the case.

(4) For a proof, see Bromwich, Infinite series (1908) (pp. 68–69), § 28

convergence is that $\Sigma \mid a_n \mid$ should converge. Similarly if a series diverges *i.e.* the series has infinity as a unique limit as explained in I and no rearrangement alters this fact the series is said to be absolutely divergent.(2)

(2) *Directions of absolute convergence and divergence*:— Any complex number can be regarded as a vector in a plane, its real and imaginary parts being the resolved parts or resolutes of the vector along the axes of reference. Instead of resolving along these axes, one might as well resolve them along any two perpendicular directions. If $a = \rho e^{i\lambda}$ and θ the angle which a st. line makes with the positive direction of the axis of reals, the resolutes in that direction and the perpendicular one are $e^{i\theta}\rho\cos(\lambda-\theta)$ and $ie^{i\theta}\rho\sin(\lambda-\theta)$. If all the terms of a series Σa_n be replaced by two such components we get two new series whose sum is the original series and each of these series is a real series multiplied by a complex factor depending on the direction of resolution. These latter do not affect the behaviour of the series and may be omitted in the discussion of convergence or divergence. Now suppose that the terms of a series be resolved in any direction and that the corresponding resolved series converges or diverges absolutely. Then that direction may be called a direction of absolute convergence or divergence(2) respectively. With this conception, the various cases that can occur may be analysed as follows:—

(i) There may be a direction of absolute divergence.

(ii) There may be two distinct directions of absolute convergence. In this case the series converges absolutely and every direction is a direction of absolute convergence.

(iii) There is only one direction of absolute convergence and no direction of absolute divergence.(5) In this case, if the series converges, the resolved part in the direction perpendicular to the direction of absolute convergence, is a conditionally convergent series and so could be rearranged by II (iii) so as to have different sums or oscillate.

(iv) There is no direction of absolute convergence or divergence. This is the case that will be examined in the sequel.

(3) *The nature of the partial sums along different directions*:—Let $a_1, a_2, \ldots$ be resolved along any direction and partial sums (in the sense explained in *III* (1)) of these resolved parts

---

(5) That absolute convergence and divergence can occur together along different directions is shown below by en example (see *III* (6) (i))

be formed; the following cases corresponding to the four discussed in *III* (2) (i)—(iv), may be distinguished:—

(i) There is a direction of absolute divergence. Using *II* (ii) we see that the partial sums have a bound in one way and no bound in the opposite way along this direction of absolute divergence,

(ii) When there are two distinct directions of absolute convergence, the partial sums along these are bounded either way by *II* (*i*) and mereover since the series converges absolutely along every direction, the partial sums are bounded either way along every direction.

(iii) If there is no direction of absolute divergence but one of absolute convergence, then the partial sums along the latter are bounded either way but the partial sums along every other direction are unbounded both ways.

(iv) When there is no direction of absolute convergence or divergence, the partial sums are unbounded both ways along every direction (in virtue of *II* (iii)). In this case the partial sums of Σa_n are said to be totally unbounded. It means that if any direction be taken along with the perpendicular one as the axes of reference, there are partial sums having arbitrarily large positive and negative x- or y- co-ordinates. Or if any st. line be drawn in the complex plane there are partial sums of Σa_n lying on either side of that line, This is the case that occurs in the hypothesis of the main theorem.

(4) *The main theorem*:—Let $a_1, a_2, a_3 \ldots$ be a sequence of complex numbers. Let $m = \underset{n \to \infty}{Lim} | a_n |$. Let there be no direction of absolute convergence or divergence for the series $\sum a_n$. Let σ be any point of the complex plane and $C_m(\sigma)$ a circle of radius km and centre σ. Then there is a particular rearrangement of Σa_n such that all its limit points lie within or on $C_m(\sigma)$. Here k is an absolute constant an upper bound for which is 10.

It is to be noted that if $m = o$, the theorem becomes that such a series could be rearranged so as to converge to σ since then $C_m(\sigma)$ is merely the point σ. Here, as in II for real series, nothing is assumed as regards the convergence of Σa_n in any rearrangement.

4 (1) The proof of the main theorem depends on a series of Lemmas(6) which depend only on the nature of a complex number as a two dimensional vector.

Lemma (i): If $\gamma = a_1 a_1 + \ldots + a_p a_p$ $(o < a_i < 1,\ i = 1, 2; p)$, then γ could be expressed by a relation of the same form in which not more than two of the co-efficients $a_1, \ldots a_p$ are different from 0 or 1.

Proof: Let $\delta = a_1 a_1 + a_2 a_2 + a_3 a_3$.

Since a_1, a_2, a_3 are three two-dimensional vectors we have a relation,

$$b_1 a_1 + b_2 a_2 + b_3 a_3 = o \text{ where the } b\text{'s are real.}$$

If one of a_1, a_2, a_3 be *0* or *1* then δ is already in the form in which not more than two are different from zero or one. So we may suppose $0 < a_i < 1$ $(i = 1, 2, 3)$. Then

$$\delta = (a_1 - b_1 x)\, a_1 + (a_2 - b_2 x)\, a_2 + (a_3 - b_3 x)\, a_3.$$

If now x be equal to the least positive or the greatest negative of the 6 numbers $\frac{a_i}{b_i}$, $\frac{a_i - 1}{b_i}$ $(i = 1, 2, 3)$ which are all different from zero, we have one of the co-efficients zero while all the others lie between *0* or *1*.

Now if in $\gamma = a_1 a_1 + \ldots + a_p a_p$ there are h co-efficients $(2 < h \leq p)$ different from zero or *1*, we may suppose that these are $a_1, \ldots, a_h$ without loss of generality. Reduce $a_1 a_1 + a_2 a_2 + a_3 a_3$ to a form in which one co-efficient is zero or one. This is possible by what has been proved above. Then in γ, h becomes $h - 1$. So we can arrive at the desired result in a finite number of steps.

Lemma (ii) If $a_1 a_1 + \ldots + a_p a_p = 0$ $(0 < a_t < 1)$ and not more than four of the co-efficients be different from *1*, then there is a similar relation (if $p > 4$) in which one of the numbers $a_1, \ldots a_p$ is absent.

We may suppose without loss of generality that a_1, a_2, a_3, a_4 are these co-efficients different from unity. If any of these be zero, the lemma is proved. So we may suppose $0 < a_i < 1$, $(i = 1, 2, 3, 4)$. Then by the previous *Lemma* $a_1 a_1 + a_2 a_2 + a_3 a_3 + a_4 a_4$ could be reduced to a form in which not more than two are different from *0* or *1*. If one of the remaining coefficients

(6) Lemmas (i) to (iv) are contained in Steinitz's paper quoted in (1) Refer: § 15 (pp. 167), Hilfssatze: 1, 2, 4 and II. Lemma (v) has no analogue in Steinitz's paper though the result is deducible from various theorems established there. Lemma (vi) is his Hilfssatze 3.

be actually 0 then the Lemma is proved. Otherwise all the co-efficients excepting two are different from 1. So we may suppose in the original relation itself that not more than two are different from 1. Let now $\gamma = a_1 a_1 + a_2 a_2 + a_3 a_3 + a_4 a_4$, where $a_i > 0$.

Since $a_1 - a_4$, $a_2 - a_4$, $a_3 - a_4$ are three complex numbers, there is a relation

$$b_1(a_1 - a_4) + b_2(a_3 - a_4) + b_4(a_3 - a_4) = 0, \quad b\text{'s real}$$

or $b_1 a_1 + b_2 a_2 + b_3 a_3 + b_4 a_4 = 0$. Where $\sum_1^4 b_i = 0$.

Therefore $\gamma = \sum_{i=1}^{4} (a_i - b_i x)\, a_i$.

Since $\sum_1^4 b_i = 0$ and all the b_i's are not zero, there are positive as well as negative b's. If x be equal to the least value of $\frac{a_i}{b_i}$ where only such b_i's as are positive are counted we see that in γ one co-efficient is zero while the others are not negative, and the sum of the co-efficients remains unaltered.

Now the original relation may without loss of generality be written as $a_1 a_1 + a_2 a_2 + a_3 a_3 + a_4 a_4 + a_5 a_5 + \ldots + a_p a_p = 0$. When $0 < a_i < 1$ $(i = 1, 2)$ while $a_i = 1$, $(2 < i < p.)$

Dividing by $\sum_1^p a_i$ we get $\sum_1^p c_i a_i = 0$ where $\sum_1^p c_i = 1$, $c_i > 0$.

Now by what has been proved above, this can be successively reduced till only 3 of $(a_1, \ldots . a_p)$ are left in which the co-efficient are all non-negative while their sum is still $\sum_1^p c_i = 1$.

We may without loss of generality suppose that this relation is $c_3 a_3 + c_4 a_4 + c_5 a_5 = 0$, $c_i \geqslant 0$ while $c_3 + c_4 + c_5 = 1$ so that $0 \leqslant c_i \leqslant 1$, $i = 3, 4, 5$.

Therefore we have

$$a_1 a_1 + a_2 a_2 + (a_3 - c_3 x) a_3 + (a_4 - c_4 x) a_4 + (a_5 - c_5 x) a_5 + a_6 a_6 + \ldots\ldots\ldots + a_p a_p = 0.$$

If now x be equal to the least among $\frac{a_3}{c_3}, \frac{a_4}{c_4}, \frac{a_5}{c_5}$ (the fraction corresponding to any $c_i = 0$ being omitted), we see that one of the the co-efficients of a_3, a_4, a_5 vanish and consequently not more than 4 are different from 1 while one of the numbers $a_1, \ldots . a_p$ is absent. Hence the Lemma is proved.

Lemma (iii). Let $a_1+a_2+a_3+\ldots\ldots+a_p=0$. Then a rearrangement $\beta_1,\ldots\beta_p$ of the a's can be found such that the successive sums $|\beta_1|$, $|\beta_1-\beta|\ldots|\beta_1+\ldots+\beta_p|$ are all less than $4m$ where m is the upper bound of the numbers $a_1,\ldots a_p$.

Proof: All the co-efficients in $a_1+\ldots+a_p=0$ are 1. Hence by *Lemma* (ii) a similar relation could be found in which one, say β_p is absent, while not more than four are different from 1 and these four lie between 0 and 1. If $p-1>4$ we can proceed in this manner and pick out $\beta_p,\ldots\beta_5$ such that if $\beta_1,\beta_2,\beta_3,\beta_4$ be these left we have $a_{k_1}\beta_1+a_{k_2}\beta_2+a_{k_3}\beta_3+a_{k_4}\beta_4+\beta_5+\ldots+\beta_k=0$ where $k=5,6,\ldots p$ and $0\leqslant a_{k_i}\leqslant 1, i<1\ldots 4$.

Hence $|\beta_1+\ldots+\beta_k|\leqslant\sum_{i=1}^{4}(1-a_{k_i})\,|\beta_1|\leqslant 4m;\ k=1,2,\ldots p$. So the Lemma is proved.(7)

Lemma (iv). If $\gamma=a_1+\ldots+a_p$ and $|\gamma|\leqslant h\,m$ where h is an integer and m the upper bound of $|a_i|$ $(i=1\ldots p)$, then a rearrangement of $a_1\ldots a_p$ say $\beta_1,\ldots\beta_p$ exists such that $|\beta_1|$, $|\beta_1+\beta_2|$, $\ldots$ $|\beta_1+\ldots+\beta_p|$ are all less than $(h+4)\,m$.

Proof: We have $a_1+\ldots+a_p-\frac{\gamma}{h}-\frac{\gamma}{h}-\ldots(h\text{ times})=0$. Therefore by *Lemma* (iii) we can rearrange these $p+h$ quantities so that the absolute value of the successive sums is less than $4m$. But each of these sums, besides the a's will involve at most h numbers each equal to $-\frac{a}{h}$. Hence if these are omitted the remaining sums are a rearrangement of $a_1,\ldots a_p$ such that their absolute values are less than $4\,m+\left|\frac{\gamma}{h}\right|h$ which is less than $(h+4)\,m$.

Lemma (v). Let a_1, a_2, $a_3\ldots$ be a sequence of complex numbers. Let there be no direction of absolute convergence or divergence for Σa_n so that the partial sums are totally unbounded *i.e.* both the half-planes into which any st. line divides the plane contain partial sums from the system Λ(8). Then the system of numbers γ where $\gamma=a_1a_1+a_2a_2+\ldots+a_pa_p$ $(0\leqslant a_i\leqslant 1, i=1,2,\ldots p)$ where $p=1,2,3\ldots$—include all points of the complex plane.

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(7) The $\beta_1,\beta_2,\beta_3,\beta_4$ which are those left over at each stage whose co-efficients may not be one, need not be the same for all $k=1,2,3,\ldots p$.

(8) Refer III (1).

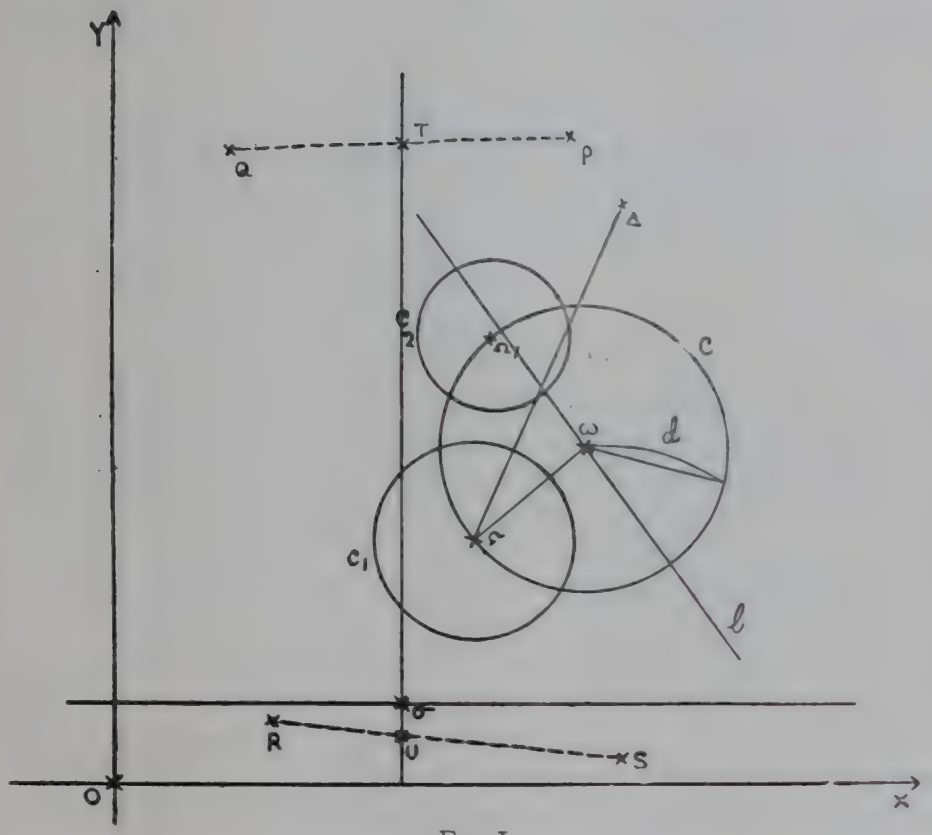

FIG. I.

Proof: This Lemma contains the kernel of the proof of the main theorem. First we may note a property of the system of numbers γ. Let γ_1 and γ_2 be any two numbers of the system. Then any point on the segment of the line joining (γ_1, γ_2) is also a point of the system. For Let

$$\left.\begin{aligned}\gamma_1 &= a_1\alpha_1 + \ldots + a_p\alpha_p \\ \gamma_2 &= b_1\alpha_1 + \ldots + b_k\alpha_k\end{aligned}\right\} \quad \left.\begin{aligned}&(0 \leqslant a_i \leqslant 1)\ (i=1,2\ldots p) \\ &0 \leqslant b_i \leqslant 1\ \ (i=1,2\ldots k)\end{aligned}\right\}$$

and say $p \geqslant k$. Let $0 < h < 1$. Then

$$h\gamma_1 + (1-h)\gamma_2 = \sum_{i=1}^{k} [a_i h + b_i(1-h)]\alpha_i + \sum_{k+1}^{p} h a_i \alpha_i$$

and since $0 \leqslant a_i h + b_i(1-h) \leqslant h + 1 - h = 1$ and $0 \leqslant h\, a_i \leqslant 1$, we see that $h\gamma_1 + (1-h)\gamma_2$ is a point of the system. But as h describes $(0, 1)$, $h\gamma_1 + (1-h)\gamma_2$ describes the segment joining γ_1 to γ_2.

Now suppose, if possible, that σ is a point not belonging to γ (*see* Fig. I). Draw parallels to the x and y axes through σ. Then at least one of the quadrants into which the plane is thus divided must contain no point of the system γ. For if P, Q, R, S are such points in the first, . . . fourth quadrant respectively, we see that PQ, RS will cut the parallel to the Y axis in T and U

which will include σ between them. But T and U are points of γ by the property proved above and so σ itself would be a point of γ which is against hypothesis. Let Q_1 be the quadrant free from points of γ. If ω be any point interior to Q_1, then the distance, d, of ω from points of γ is a positive number. Let C be a circle of radius d and centre ω. Then C will have on its circumference at least one point Ω which is either a point of γ or a limit point of points of γ. There cannot be another such point Ω_1; for if there were, we could draw small circles C_1 and C_2 round Ω and Ω_1 such that the line joining any two points, one in each of these circles, will always cut C. Since C_1 and C_2 will contain points of γ however small their radii be, it would follow that C contains points of γ which is impossible since d is the lower bound of the distances of ω from points of γ. So there cannot be another point Ω_1. Let now l be a line perpendicular to the line $(\Omega\omega)$ and passing through ω. Then there cannot be a point of γ on that side of the line l which does not contain Ω. For if Δ is such a point, we can draw a circle C_1 round Ω so small that the join of any point inside C_1 to Δ cuts C and since C_1 contains points of γ however small its radius be, it would follow as above that C contains a point of γ which is again impossible. Hence there cannot be any such point. But this contradicts the hypothesis of total unboundedness of the partial sum of Λ since obviously any partial sum in Λ is a point of the system γ. Hence there cannot be such a point as σ. So the system γ include all points of the plane.

Lemma (vi). Let $a_1, a_2, \ldots\ldots$ be a sequence satisfying the condition of *Lemma* (v). Then given any point σ in the complex plane, there exists a partial sum A from Λ such that $|\sigma - A| \leq 2m$ where m is the upper bound of $|a_i|$ $(i=1, 2\ldots.)$. Further this partial sum could be so chosen as to include any given number from $a_1, a_2\ldots\ldots$

Proof: By *Lemma* (v), there is a p such that

$$\sigma = a_1\alpha_1 + \ldots\ldots + a_p\alpha_p \quad (0 \leq a_i \leq 1).$$

By *Lemma* (i) there is a similar relation for σ in terms of $(a_1, \ldots a_p)$ in which not more than two co-efficients differ from zero or one. If these two numbers whose co-effieients are not zero or one be omitted, the remaining is a partial sum A of Λ and we have

$$|\sigma - A| \leq 2m.$$

If any particular element a_i say, should be included, we omit it from the original sequence. The rest $a_2, a_3\ldots$ also satisfy the condition of the *Lemma*. And so there is a partial sum A_1

such that $|(\sigma-a_1)-A^1| \leqslant 2m$. If $A=a_1+A^1$ we get the desired results

(4·2). *Proof of the main theorem*: By hypothesis, the sequence $a_1, a_2, \ldots$ satisfy the condition of *Lemma* (v) or (vi). If a finite number of the a's be omitted, the remaining also satisfy the condition. Let m_k be the upper bound of $|a_k|, |a_{k+1}|, \ldots$ so that $m_k \to m$ as $k \to \infty$ and $m_k \geqslant m_{k+1}$.

In order to avoid the multiplicative axiom, we arrange the partial sums of Λ as follows. Let A be any partial sum. If a_k occurs in it, substitute 2^k in its place and add up for all the terms occurring in A. The result is an integer which may be associated with A and be called the rank of A. Since any integer could be expressed in one and only one way in the form $a_0 2^p + \ldots\ldots + a_p$ where the a's are o or 1, every partial sum has only one rank and every integer is the rank of one and only one of the partial sums of Λ. Thus we may properly speak of the n^{th} partial sum.

By *Lemma* (vi) we can final a partial sum A which includes a_1 such that $|\sigma - A| \leqslant 2\, m_1$ Let π_1 be the first partial sum having this property. Then $|\sigma - \pi_1| \leqslant 2\, m_1$. Let, now, the terms of π_1 be omitted from $a_1, a_2, \ldots$ and let the partial sums of the remaining a's be arranged as before. The upper bound of the remaining terms is less than or equal to m_2 and let π_2 be the first partial sum which contains the first term of these remaining a's and has the property $|(\sigma-\pi_1)-\pi_2| \leqslant 2\, m_2$. In this way we can continue and arrive at the result that $|\sigma - (\pi_1 + \ldots + \pi_k)| \leqslant 2\, m_k$ where π_k is the first partial sum which includes the first term of the a's when the terms contained in $\pi_1, \ldots \pi_{k-1}$ are omitted from them. Now

$$|\pi_k| = \left|\left\{\sigma-(\pi_1+\ldots+\pi_{k-1})\right\} - \left\{\sigma-(\pi_1+\ldots+\pi_k)\right\}\right|$$
$$\leqslant 2\, m_{k-1} + 2\, m_k \leqslant 4\, m_{k-1}.$$

Let π_k contain p_k terms say $a_{k_1}, a_{k_1}, \ldots a_{kp_k}$. Since $|\pi_k| \leqslant 4\, m_{k-1}$ we can by *Lemma* (iv) so rearrange these a's such that the successive sums are all less than $8\, m_{k-1}$ in absolute value. Here also we may consider the $p_k!$ rearrangements and take the first which has the property stated. We may suppose that terms of π_k so rearranged are $a_{k_1} + a_{k_2} + \ldots\ldots + a_{kp_k}$. Then we obtain a series

$$a_{1_1}+\ldots\ldots+a_{1_{f_1}}+a_{2_1}+\ldots\ldots+a_{2_f}+\ldots\ldots+a_{k_1}+\ldots+a_{k_{f_k}}+\ldots$$

where if S_n denotes the sum to n terms and $k_1 \leqslant n \leqslant k_{f_k}$,

$$|\sigma - S_n| \leqslant |\sigma-(\pi_1+\ldots+\pi_{k-1})| + |a_{k_1}+a_{k_2}+\ldots+a_n|$$
$$\leqslant 2\,m_{k-1}+8\,m_{k-1}=10\,m_{k-1}.$$

And as $n\to\infty$, $k\to\infty$ and $m_{k-1}\to m$ we have

$\overline{\underset{n\to\infty}{Lim}}\,|\sigma-S_n| \leqslant 10\,m$ from which the result stated in the main theorem follows.

(5) *The case* $m=0$:—If $m=0$, that is the terms of the series form a null sequence, the main theorem states that the series, under the conditions of the theorem, could be rearranged so as to converge to any number σ in the complex plane. This result enables us to give a complete classification of complex series whose terms tend to zero:—

(i) There is a direction of absolute divergence in any particular rearrangement. Then however rearranged this direction will continue to be one of absolute divergence and the complex series has the unique limit infinity. The complex series is absolutely divergent.[9]

(ii) There is no direction of absolute divergence[9] but two distinct directions of absolute convergence. Here the series converges absolutely, every direction is one of absolute convergence and the series has a unique finite limit however rearranged.

(iii) There is no direction of absolute divergence[10] but one and only one direction of absolute convergence. Here for all rearrangements this direction of absolute convergence is preserved, and since there is no direction of absolute divergence and no second direction of absolute divergence, the complementary part of the series perpendicular to this direction of absolute convergence must be a real series (except for a constant complex factor) which falls under case (iii), II. So this part could be rearranged so as to have any number in that direction as sum. Hence the limit points of the series for all rearrangement lie on a st. line perpendicular to the direction of absolute convergence at a distance along the latter direction equal to the sum of the absolutely convergent part. Here it is to be noted that since no rearrangement whatever affects the absolutely convergent part, not only convergent rearrangements of the complex series but also all possible rearrangements will have all their limit points on the

(9) This is excluded by the hypothesis of absolute convergence in two directions and so is superfluous.

(10) *See* foot-note (5).

st. line specified. In this case the series may be said to have a st. line sum.

(iv) There is no direction of absolute convergence or divergence. This is the case occurring in the hypothesis of the main theorem and here $m=0$. Consequently the complex series could be rearranged so as to converge to any number in the complex plane. It is unnecessary to state that for all rearrangements the limit points lie in the complex plane. What is to be noted is that each point is the sum of a particular rearrangement of the terms of the given sequence. In this case the series may be said to have a plane sum.

These four cases are mutually exclusive and thus give a comprehensive classification of series formed out of a null sequence.

(5·1) If $m\neq 0$ but finite, the main theorem states that there is a particular rearrangement of the series all of whose limit points lie within a circle of radius k_m round any given point σ of the complex plane. Here the hypothesis is that there is no direction of absolute convergence or divergence. Proceeding to the general case when $m\neq 0$, *firstly* it is evident that the series can never be rearranged so as to have a unique finite limit. *Next* it may happen that finite limit points lie on a st. line. But we cannot assert that the limit points fill up the plane. All the same, if all the limit points lie on a st. line it is obvious that the direction perpendicular to this line is one of absolute convergence and conversely. In this case the limit points may or may not fill up the whole line. *Thirdly* the limit points may not be confined to a single line. In this case if there are finite limit points, there cannot be a direction of absolute convergence or divergence and so by the main theorem the limit points must be distributed all over the plane and cannot be confined to even a finite number of st. lines. But here also we cannot assert that the limit points fill up the plane, They may or may not. Thus an exhaustive classification of the type given for $m=0$ is impossible when $m\neq 0$. Examples to illustrate the various possibilities are given below. It is also shown that in some cases when $m=\infty$, the main theorem may be utilised by proper grouping of the terms of the series to give some information regarding their limit points.

(6) (i) Example to show that directions of absolute convergence and divergence may occur together:—Let $a_n = 1+\frac{i}{n^2}$.

The real part $1+1+1+\ldots\ldots$ is absolutely divergent while the other $\Sigma \frac{1}{n^2}$ is absolutely convergent.

(ii) Example to show that the limit points, when $m \neq 0$ may all lie on a line but need not fill it:—$a_n = (-1)^{n-1} + \frac{i}{n^2}$. Here $m=1$. Also $\Sigma \frac{1}{n^2}$ is absolutely convergent and all the limit points lie on the line $y = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. But the real part is $1-1+1-1+\ldots\ldots$ If in any particular rearrangement S_n, the sum to n terms, contain p_n and q_n positive and negative terms, $S_n = p_n - q_n$ and so S_n can take only integral values or the value zero. And by properly choosing p_n and q_n we can make any given integer a limit point of S_n. Hence the limit points are $\lambda + \frac{\pi^2}{6} i$ where λ is an integer and so do not fill up the whole line $y = \frac{\pi^2}{6}$.

(iii) Example to show that limit points may lie on a line and may fill up the whole line:—Let $a_{2n-1} = 1 + \frac{i}{(2n-1)^2}$, $a_{2n} = -\theta + \frac{i}{(2n)^2}$ where θ is a positive irrational number. Here also all the limit points lie on the line $y = \frac{\pi^2}{6}$. But the real part is $1-\theta+1-\theta+\ldots\ldots$ and using the notation of the previous example $S_n = p_n - q_n \theta$. But if θ is irrational it is known that by proper choice of the sequence p_n and q_n, $p_n - q_n \theta$ may be made to tend to any real number as limit. Hence all points of the line $y = \frac{\pi^2}{6}$ are limit points.

(iv) Example to show that the limit points do not lie on a line but do not fill up the plane:—$a_n = (-1)^{n-1}\left(1 + \frac{i}{n}\right)$. Here $m=1$. There is no direction of absolute convergence or divergence. For the resolved part along the direction θ is, but for a constant factor, equal to $\Sigma (-1)^{n-1}\left(\cos\theta + \frac{i \sin\theta}{n}\right)$. If $\theta \neq \frac{\pi}{2}$, the positive and negative parts diverge separately and so there are not directions of absolute divergence or convergence.

If $\theta = \frac{\pi}{2}$, $\Sigma \frac{(-1)^{n-1}}{n}$ converges but not absolutely. Hence there is no direction of absolute convergence or divergence. Hence by the main theorem the limit points do not all lie on a line but are distributed all over the plane. But the real part is $1-1+1-1+\ldots$ which can only have an integer as a limit point. Hence all the limit points lie on the lines $x=\lambda$ where λ is zero or an integer and so do not fill up the whole plane.

(v) Example to show that the limit points even if $m \neq 0$, may fill up the whole plane:—

First let $a_n = (-1)^{n-1}\left(\frac{1}{n} + \frac{i}{\log n}\right)$. Here $|a_n| \to 0$ and it is easily verified as in (iv) that there is no direction of absolute convergence or divergence. Hence by III (5) (iv), the series could be rearranged so as to have any point σ for its sum. Let now $\beta_n - \beta_{n-1} = (-1)^{n-1}\left(\frac{1}{n} + \frac{i}{\log n}\right) = a_n$. Then $\beta_n = 1 - \left(\frac{1}{2} + \frac{i}{\log 2}\right) + \ldots\ldots + (-1)^{n-1}\left(\frac{1}{n} + \frac{i}{\log n}\right)$ if $\beta_1 = 1$. So β_n tends to a finite limit which is obviously not zero. Consider now the series $C_1 + C_2 + \ldots\ldots + C_n \ldots\ldots$ where $C_{2n} = -\beta_n$, $C_{2n-1} = \beta_{n+1}$ so that $C_{2n-2} + C_{2n-3} = a_n$. Now writing ΣC_n in the form $\Sigma (C_{2n-3} + C_{2n-2}) = \Sigma (\beta_n - \beta_{n-1}) = \Sigma a_n$ and rearranging Σa_n so as to converge to σ we see that if now a_n be replaced by $C_{2n-3} + C_{2n-2}$, a rearrangement of ΣC_n is obtained for which σ is a limit point and since $\overline{\lim_{n\to\infty}} |C_n|$ is the same as $\overline{\lim_{n\to\infty}} |\beta_n|$ which is not zero we have the desired example.

(vi) By the method of grouping employed in example (v), we can sometimes obtain information regarding the limit points of series whose terms are not bounded. For example, consider the series

$$1-2+3-4+5-6+\ldots.$$

Here $a_n = (-1)^{n-1} n$ and $|a_n| = n \to \infty$ as $n \to \infty$. Let $\beta_{2n-1} = a_{3n-2} + a_{3n-1}$ and $\beta_{2n} = a_{6n-3} + a_{5n}$, $n = 1, 2, \ldots\ldots$

Then $\beta_{2n-1} = (-1)^n$, $\beta_{2n} = -3$. Here $m=3$ for the series $\Sigma \beta_n$. Let in any rearrangement of $\Sigma \beta_n$, there be p positive and q

negative terms in S_n, the sum to first n terms. Let there be further r ones and s threes in q. Then $S_n = p - (\gamma + 3s)$. By proper choice of the sequence p, r, s, S_n may be made to have any integer for its limit and if in that rearrangement the β's be replaced by α's we have a rearrangement of the original series for which any given integer is a limit point. Further for the given series there cannot be any other limit points than integers. Hence $1-2+3-4+5-6+\ldots\ldots$ could be rearranged so as to have any integer for one of its limits. Hence given any integer, it can be expressed in an infinity of ways as the difference of two sums, one sum those of even and the other those of odd integers, the numbers of these odd and even integers exceeding any given limit whatever.

GENERALISATION OF JACOBI'S θ—FUNCTION FORMULAE.

By K. Venkatachaliengar.

Jacobi's well known formulae on the multiplication of θ—functions can be generalised by making use of general orthogonal linear substitution. The generalised theorem runs as follows:—

If variables $(l, m, n, \ldots\ldots p)$ and $(l_1, m_1, n_1, \ldots\ldots p_1)$ are connected by means of the relations

$$\text{(i)} \begin{cases} a_{1_1} l + a_{1_2} m + \ldots\ldots a_{1_p} p = a_{1_1} l_1 + a_{2_1} m_1 + \ldots a_{p_1} p_1 \\ a_{2_1} l + a_{2_2} m + \ldots\ldots a_{2_p} p = a_{1_2} l_1 + a_{2_2} m_1 + \ldots a_{p_2} p_1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{p_1} l + a_{p_2} m + \ldots\ldots a_{p_p} p = a_{1_p} l_1 + a_{2_p} m_1 + \ldots a_{p_p} p_1 \end{cases}$$

where $a_{i_k} = -a_{k_i}$. Then the relation between the sets of variables is orthogonal. Moreover let the coefficients involved be all integers positive or negative. Let k be the value of the determinant

$|a_{1_1}, a_{2_2} \ldots a_{p_p}|$ and let $k \neq 2$ and let the orthogonal relation be written in the form

$$\begin{aligned} l &= a_{1_1} l_1 + a_{2_1} m_1 + a_{3_1} n_1 + \ldots\ldots + a_{p_1} p_1 \\ m &= a_{1_2} l_1 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{p_2} p_1 \\ p &= a_{1_p} l_1 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{p_p} p_1 \end{aligned}$$

Then let the complete systems of solutions of the set of linear equations

$$\text{(A)} \quad a_{r_1} l + a_{r_2} m + a_{r_3} n + \ldots a_{r_p} p = o \pmod{k}$$

$r = 1, 2, 3 \ldots\ldots p$, be given by

$(\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_p)$, $(\mu_1, \mu_2, \ldots \mu_p)$ etc., where

$l = \lambda_1 \pmod{k}$, $m = \lambda_2 \pmod{k}$, $\ldots p = \lambda_p \pmod{k}$.

and

$$l = \mu_1 \pmod{k}, \quad m = \mu_2 \pmod{k}, \ldots p = \mu_p \pmod{k}$$

and so on.

And let the corresponding systems of solutions of

(B) $a_{1_r} l_1 + a_{2_r} m_1 + a_{3_r} n_1 + \ldots + a_{p_r} p_1 = o \ [mod\ k]$

$r = 1, 2, \ldots p$, be given by

$(\lambda'_1, \lambda'_2, \lambda'_3 \ldots \lambda'_p)$, $(\mu'_1, \mu'_2, \mu'_3 \ldots \mu'_p)$ and etc.,

and let

$$f_{\lambda_1}(x) = \sum_{r=-\infty}^{r=+\infty} q^{l^2/k^2} e^{\frac{2ilx}{k}}, \quad f_{\lambda_2}(x) = \sum_{r=-\infty}^{r=+\infty} q^{m^2/k^2} e^{\frac{2imx}{k}},$$

and etc.,

where $l = rk + \lambda_1$, $m = rk + \lambda_2$, ... and etc.,

and $q = e^{i\pi\tau}$ where the imaginary part of τ is $+ve$ and let

$$x_1 = a_{1_1} x + a_{2_1} y + a_{3_1} z + \ldots\ldots$$

$$y_1 = a_{1_2} x + a_{2_2} y + a_{3_2} z + \ldots\ldots$$

$$z_1 = a_{1_3} x + a_{2_3} y + a_{3_3} z + \ldots\ldots$$

. . . . . .

then my generalisation takes the form

$$\sum_{(\lambda, \mu, \ldots\ldots)} f_{\lambda_1}(x) \ f_{\lambda_2}(y) \ldots\ldots\ldots = \sum_{(\lambda', \mu' \ldots\ldots)} f_{\lambda'_1}(x_1) \ f_{\lambda'_2}(y) \ldots\ldots$$

and it is also shown in the paper how each of these functions can be expressed in terms of θ-functions. One of the particular results takes the very elegant form viz., if $x_1 = \frac{1}{3}(-x+2y+2z)$, $y_1 = \frac{1}{3}(2x-y+2z)$, $z_1 = \frac{1}{3}(2x+2y-z)$,

then

$$\sum_{r=o}^{2} q^{r^2/3} e^{2rix} \theta_3\left(x + \frac{\pi r\tau}{3}\right) \theta_3\left(y + \frac{\pi r\tau}{3}\right) \theta_3\left(z + \frac{\pi r\tau}{3}\right)$$

remains unaltered by the substitution of x_1, y_1, z_1 in the place of x, y, z.

Now first of all it is necessary to show that the substitution (i) is orthogonal.

Now multiply the equations by the minors $[A_{1_1}, A_{2_1}, \ldots A_{p_1}]$ respectively of $[a_{1_1}, a_{2_1}, \ldots\ldots a_{p_1}]$ respectively in the determinant $|a_{1_1}, a_{2_2}, \ldots\ldots a_{p_p}|$ and adding we obtain

$$kl = \left(a_{1_1} A_{1_1} + a_{1_2} A_{2_1} + \ldots\ldots a_{1_p} A_{p_1}\right) l_1$$

$$+ \left(a_{2_1} A_{1_1} + a_{2_2} A_{2_1} + \ldots\ldots a_{2_p} A_{p_1}\right) m_1 \ldots\ldots$$

$$=(2a_{1_1}A_{1_1}-k)\,l_1+2a_{2_2}A_{2_1}\,m_1+2a_{3_3}A_{3_1}n_1+\ldots+2a_{p_p}A_{p_1}p_1$$

with corresponding expressions for $m, n, \ldots . p$ and if we multiply by $(A_{1_1}, A_{1_2}, \ldots . A_{1_p})$ and add we obtain similarly $kl_1 = (2a_{1_1}\,A_{1_1}-k)\;l+2\,a_{2_2}\,A_{1_2}m\ldots .2\,A_{1_p}\,p\,a_{p_p}$ and with corresponding expressions for $m_1, n_1, \ldots\ldots p_1$. Therefore the relation is orthogonal.

Let k be an integer other than 2 and all the a's denote positive or negative integers less than k in absolute value.

Let the orthogonal relation obtained in the preceding paragraph be written in the form

$$l = a_{1_1}\;l_1+a_{2_1}\,m_1+\ldots . a_{3_1}\,n_1+\ldots\ldots$$
$$m = a_{1_2}\;l_1+a_{2_2}\,m_1+\ldots . a_{3_2}\,n_1+\ldots .$$
$$\begin{array}{cccccc} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array}$$

Now, let $l_1, m_1, n_1 \ldots\ldots$ and etc. take the complete sets of integral values positive or negative satisfying the relations

$$(A)\begin{cases} a_{1_1}\,l+a_{1_2}\,m+\ldots\ldots a_{1_p}\,p = o \pmod k \\ a_{2_1}\,l+a_{2_2}\,m+\ldots\ldots a_{2_p}\,p = o \pmod k \\ \quad .\qquad .\qquad\qquad . \\ \quad .\qquad .\qquad\qquad . \\ \quad .\qquad .\qquad\qquad . \\ a_{p_1}\,l+a_{p_2}\,m+\ldots\ldots a_{p_p}\,p = o \pmod k \end{cases}$$

and then $(l_1, m_1, n_1 \ldots .)$ will also be integers satisfying

$$(B)\begin{cases} a_{1_1}\,l_1+a_{2_1}\,m_1+\ldots\ldots a_{p_1}\,p_1 = o \pmod k \\ a_{1_2}\,l_1+a_{2_2}\,m_1+\ldots\ldots a_{p_2}\,p_1 = o \pmod k \\ \quad .\qquad .\qquad\qquad . \\ \quad .\qquad .\qquad\qquad . \\ \quad .\qquad .\qquad\qquad . \\ a_{1_p}\,l_1+a_{2_p}\,m_1+\ldots\ldots a_{p_p}\,p_1 = o \pmod k \end{cases}$$

and *vice versa.*

It is also obvious that if $(l, m, n \ldots .)$ take all integral values positive or negative consistent with relations (A) then $(l_1, m_1, n_1, \ldots .)$ take all the integral values consistent with relations (B) and *vice versa.* And corresponding to one set of values $(l, m, n \ldots . p)$ there is one and only one set of values $(l_1, m_1, n_1 \ldots . p_1)$.

Let the complete system of values $(l, m, n, \ldots p)$ satisfying the relations (A) be denoted by $(\lambda_1, \lambda_2, \lambda_3 \ldots)$, $(\mu_1, \mu_2, \mu_3 \ldots)$ and etc., where

$$l \equiv \lambda_1 \pmod k; \quad m \equiv \lambda_2 \pmod k \ldots\ldots \quad p \equiv \lambda_p \pmod k$$

$$l \equiv \mu_1 \pmod k; \quad m \equiv \mu_2 \pmod k \ldots\ldots p \equiv \mu_p \pmod k$$

and the complete system of values consistent with the relations (B) be denoted by the same letters with dashes, viz., $(\lambda'_1, \lambda'_2 \lambda'_3 \ldots \lambda'_p)$, $(\mu'_1, \mu'_2, \mu'_3 \ldots \mu'_p) \ldots\ldots$ etc.

Next consider the functions given by

$$f_{\lambda_1}(x) = \sum_{r=-\infty}^{r=+\infty} q^{l^2/k^2} e^{\frac{2ilx}{k}}, \quad f_{\lambda_1}(x) = \sum_{r=-\infty}^{r=+\infty} q^{m^2/k^2} e^{\frac{2imk}{k}} \text{ and etc.}$$

where $l = rk + \lambda_1$, $m = rk + \lambda_2$, $n = rk + \lambda_3 \ldots\ldots$ and $kq = e^{i\pi\tau}$ where the imaginary part of τ is positive then it is easily seen that the series considered are absolutely convergent. Hence by Cauchy's theorem we can multiply any number of such series and rearrange them in any order we like. Now

$$(a) \ldots\ldots \sum f_{\lambda_1}(x) \; f_{\lambda_2}(y) \; f_{\lambda_3}(z) = \sum_{r=-\infty}^{r=+\infty} \sum \sum \sum \ldots q^{\frac{l^2+m^2+n^2\ldots p^2}{k^2}} e^{\frac{2i}{k}\sum lx}$$

where the sign of summation on the left represents that it has to run through $(\lambda, u, \ldots\ldots$ etc.)

Take new variables $(x, y, z, \ldots)$ which are connected with $(x_1, y_1, z_1 \ldots)$ by the relations

$$x_1 = a_{1_1} x + a_{2_1} y + a_{3_1} z + \ldots.$$

$$y_1 = a_{1_2} x + a_{2_2} y + a_{3_2} z + \ldots.$$

$$z_1 = a_{1_3} x + a_{2_3} y + a_{3_3} z + \ldots.$$

Now by the property of the orthogonal determinant if one set of variables $(l, m, n, \ldots p)$ corresponds to $(l_1, m_1, n_1 \ldots\ldots)$. $\sum l^2 = \sum l_1^2$ and $\sum l x = \sum l_1 x$.

Now consider

$$\sum f_{\lambda'_1}(x_1) \; f_{\lambda'_2}(y_1) \; f_{\lambda'_3}(z_1) \ldots\ldots\ldots$$

$$= \sum \sum \sum q^{\frac{\sum l_1^2}{k^2}} e^{2i \sum l_1 x_1}$$

where the sign of summation on the left runs through $\lambda', \mu' \ldots\ldots$ etc.

We have already proved that corresponding to one set of variables (l, m, n....etc.); there corresponds one and only one set of values (l_1, m_1, n_1....etc.) consistent with relations (A) and (B) respectively. Hence, to each term of the series of the right hand side of (a) there corresponds one term which is equal to it in a series on the right hand side of (b) and *vice versa*. And therefore the two series considered are identical. Hence

$$\sum_{(\lambda, \mu, \ldots \text{etc.})} f_{\lambda_1}(x)\, f_{\lambda_2}(y)\, f_{\lambda_3}(z) \cdots = \sum_{(\lambda', \mu', \ldots \text{etc.})} f_{\lambda'_1}(x_1)\, f_{\lambda'_2}(y_1)\, f_{\lambda'_3}(z_1) \ldots$$

This is the extension of Jacobi's formulae. For each of the functions f can be expressed in terms of Θ functions in the following way. [The notation used here is that used in Whittaker and Watson's Modern Analysis.]

$$\sum_{-\infty}^{+\infty} q^{\left(\frac{rk+\lambda}{k}\right)^2} e^{2ix\left(r+\frac{\lambda}{k}\right)} = \left[q^{\lambda^2/k^2}\right] e^{\frac{2ix\lambda}{k}} \sum_{-\infty}^{+\infty} q^{r^2} e^{2ir\left[x+\frac{\pi t\lambda}{k}\right]}$$

$$= q^{\lambda^2/k^2} e^{\frac{2ix\lambda}{k}} \Theta_3\left(x+\frac{\pi t\lambda}{k}\right)$$

Hence the theorem enunciated above becomes

$$\sum_{(\lambda, \mu, \ldots)} q^{\frac{\sum \lambda_1^2}{k^2}} e^{\frac{2i}{k}\left\{\lambda_1 x+\lambda_2 y \ldots\right\}} \cdot \Theta_3\left(x+\frac{\pi t\lambda_1}{k}\right)\Theta_3\left(y+\frac{\pi t\lambda_2}{k}\right)$$

$$= \sum_{(\lambda', \mu' \ldots)} q^{\frac{\Sigma\lambda_1'^2}{k^2}} e^{\frac{2i}{k}\left\{\lambda_1' x+\lambda_2' y \ldots\right\}} \Theta_3\left(x_1+\frac{\pi t\lambda'_1}{k}\right)\Theta_3\left(y_1+\frac{\pi t\lambda'_2}{k}\right)$$

This theorem is of theoretical interest only and becomes complicated in the general case. I have discussed here a simple, symmetrical transformation which bears a resemblance to Jacobi's theorem.

Let

$$x_1 = \tfrac{1}{3}[-x+2y+2z]$$
$$y_1 = \tfrac{1}{3}[2x-y+2z]$$
$$z_1 = \tfrac{1}{3}[2x+2y-z]$$

and let (l, m, n) be connected with (l_1, m_1, n_1) by the same relations. Now this relation is orthogonal and as $(l_1-m_1) = -(l-m)$ and so on, we get that if (l, m, n) runs through all the integral values $+ve$ or $-ve$ which are such that $(l-m) = o \pmod 3$ and $(m-n) = o \pmod 3$, then (l_1, m_1, n_1) run

through the same system of values as (l, m, n). Hence the theorem becomes

$$\sum_{r=0}^{r=2} f_r(x) f_r(y) f_r(z) = \sum_{r=0}^{r=2} f_r(x)(y_1) f_r(z_1)$$ or written in terms of Θ functions it becomes

$$\sum_{r=0}^{r=2} q^{r^2/3} e^{2ri\Sigma x} \Theta_3\left(x + \frac{\pi r l}{3}\right) \Theta_3\left(y + \frac{\pi r l}{3}\right) \Theta_3\left(z + \frac{\pi r l}{3}\right)$$

$$= \sum_{r=0}^{r=2} q^{r^2/3} e^{2ri\Sigma x} \Theta_3\left(x_1 + \frac{\pi r l}{3}\right) \Theta_3\left(y_1 + \frac{\pi r l}{3}\right) \Theta_3\left(z_1 + \frac{\pi r l}{3}\right)$$

In a similar manner one can discuss the transformation

$$x_1 = \frac{1}{n}\{-x + (n-1)\, y + (n-1)\, z + \ldots \text{ etc.} \ldots\}$$

$$y_1 = \frac{1}{n}\{(n-1)\, x - y + (n-1)\, z + \ldots \text{ etc.} \ldots\}.$$

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In another paper I propose to consider further examples of third order transformations and the relations obtained from them in the case of the Jacobian elliptic functions and σ functions.

ON THE EXPANSION OF BESSEL FUNCTIONS IN A SERIES OF MATHIEU FUNCTIONS AND ON A PROPERTY OF MATHIEU FUNCTIONS.

BY N. A. SHASTRI.

I. *Introduction.*

The expansions of Mathieu Functions in terms of Bessel Functions have been investigated by many writers, notable amongst them being Dr. Dhar † Dr. J. Dougall* and Goldstein,‡ but uptill now no writer has investigated the problem of expressing Bessel Functions in terms of Mathieu Functions. This latter method has its own advantages in as much as, it will enable us to obtain some of the properties of Mathieu Functions. Sections 2—4 deal with preliminary Lemmas and the main problem is discussed in sections 5—7. In the last section is obtained the relation between $ce_0(\theta, q)$, and $ce_2(\theta, q)$.

II. Let us first discuss the uniform convergence of the series,

$$\sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2}kh\cos\theta)^{\mu+\nu+2m}}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \Gamma(\mu+\nu+2m+1) \quad \ldots\ldots(1)$$

μ and ν being positive integers.

Consider the series,

$$\sum_{m=0}^{\infty} \frac{(\frac{1}{2}kh)^{\mu+\nu+2m}}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \Gamma(\mu+\nu+2m+1) \quad \ldots\ldots(2)$$

It can be very easily shown that for this series,

$$\frac{u_{n+1}}{u_{n+2}} = \frac{2n^2}{(kh)^2}\left[1+O(\tfrac{1}{n^2})\right]$$

$$\therefore \operatorname*{Lt}_{n\to\infty} \frac{u_{n+1}}{u_{n+2}} = \infty.$$

$\therefore$ The series (2) is convergent. Again $(-)^m (\cos\theta)^{\mu+\nu+2m}$ is always less than unity.

Therefore the series (1) is uniformly convergent †.

† Dhar: The Journal of the Ind. Math. Soc. Vol. XVI.
* Dougall; Proc. Edin. Math. Soc. Vol. XXXIV.
‡ Goldstein; Trans. Camb. Phil. Soc. Vol. XXIII, No. XI.
† Bromwich—Infinite Series, Chap. VII.

III. The value of $\int_0^{2\pi} J_\mu(kh\cos\theta)\, J_\nu(kh\cos\theta)\, d\theta$ will now be investigated.

Now*

$$J_\mu(kh\cos\theta)\, J_\nu(kh\cos\theta) =$$

$$\sum_{m=0}^{\infty} \frac{(-)^m \left(\tfrac{1}{2} kh\cos\theta\right)^{\mu+\nu+2m} \Gamma(\mu+\nu+2m+1)}{m!\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)}.$$

Now the series on the right hand side is proved to be uniformly convergent in II and hence it can be integrated term by term. Therefore,

$$\int_0^{2\pi} J_\mu(kh\cos\theta)\, J_\nu(kh\cos\theta)\, d\theta =$$

$$\sum_{m=0}^{\infty} \frac{(-)^m \Gamma(\mu+\nu+2m+1)\left(\tfrac{1}{2} kh\right)^{\mu+\nu+2m}}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \times$$

$$\times \int_0^{2\pi} (\cos\theta)^{\mu+\nu+2m}\, d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-)^m \Gamma(\mu+\nu+2m+1)\left(\tfrac{1}{2} kh\right)^{\mu+\nu+2m}}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \cdot$$

$$2\left[1+(-1)^{\mu+\nu}\right] \int_0^{\frac{\pi}{2}} (\cos\theta)^{\mu+\nu+2m}\, d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-)^m \Gamma(\mu+\nu+2m+1)\left(\tfrac{1}{2} kh\right)^{\mu+\nu+2m}}{\Gamma(m+1)\,\Gamma(\mu+\nu+m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)} \cdot$$

$$\frac{2\left[1+(-1)^{\mu+\nu}\right]\sqrt{\pi}}{2}\,\Gamma\left(\frac{\mu+\nu+2m+1}{2}\right)$$

$$\Gamma\left(\frac{\mu+\nu+2m+2}{2}\right)$$

$$= \begin{cases} 0 \text{ if } \mu+\nu \text{ is odd.} \\ \sum_{m=0}^{\infty} \left[\dfrac{2(-)^m (kh)^{\mu+\nu+2m} \left\{ \Gamma\left(\dfrac{\mu+\nu+2m+1}{2}\right) \right\}^2}{\Gamma(m+1)\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)\,\Gamma(\mu+\nu+m+1)} \right] \\ \qquad\qquad \text{if } \mu+\nu \text{ is even.} \end{cases}$$

$= 2\beta_{\mu,\nu}$ (say) if $\mu+\nu$ is even.

It is very easy to show that the series represented by $\beta_{\mu,\nu}$ is convergent and so $\beta_{\mu,\nu}$ is finite.

* Watson—Bessel Functions, Chap. V.

IV. Consider the series $\sum_{r=0}^{\infty} \phi_{2n}(r)\, J_{2r}(kh\cos\theta)$, where $\phi_{2n}(r)$ occurs as the co-efficient of $\cos 2r\theta$ in

$$\dagger\, c\, e_{2n}(\theta, q) = a_{0,n} \sum_{r=0}^{\infty} \phi_{2n}(r)\cos 2r\theta.$$

Now $a_{0,n}\ \phi_{2n}(r)$ is bounded for small values of q.

$$\therefore \quad |\, a_{0,n}\cdot\ \phi_{2n}(r)\,| < K.$$

Hence. $\sum_{r=0}^{\infty} \phi_{2n}(r)\ \ J_{2r}(kh\cos\theta) < K \sum_{r=0}^{\infty} J_{2r}(kh\cos\theta).$

Again $J_{2r}(kh\cos\theta) < \sum_{m=0}^{\infty} \left(\frac{kh}{2}\right)^{2m+2r} / (2r+m)!\ 2m! = H_{2r}(kh)$

say,

$$\therefore \sum_{r=0}^{\infty} \phi_{2n}(r)\ \ J_{2r}(kh\cos\theta) < K \sum_{r=0}^{\infty} H_{2r}(kh).$$

Now it can be very easily shown that

$$\frac{H_{2r}(kh)}{H_{2r+2}(kh)} = \frac{2^4 - r^2}{(kh)^2}\left[1 + O\left(\frac{1}{r^2}\right)\right].$$

$$\therefore \operatorname*{Lt}_{r\to\infty} \frac{H_{2r}(kh)}{H_{2r+2}(kh)} > 1.$$

Hence $\sum_{r=0}^{\infty} H_{2r}(kh)$ is convergent. Therefore both the series

$\sum_{r=0}^{\infty} \phi_{2n}(r)\ \ J_{2r}(kh\cos\theta)$ and $\sum_{r=0}^{\infty} (-)^r\, \phi_{2n}(r)\ \ J_{2r}(kh\cos\theta)$

are uniformly convergent.

V. The expansion of $ce_{2r}(\theta, q)$ in terms of Bessel's Functions have been found out by Dr. Dhar† and Goldstein* by two methods (i) by using integral equations (ii) by transforming the Mathieu's equation into Bessel's differential equation by suitable substitutions. In this section the possibility of expanding $J_{2n}(kh\cos\theta)$ in terms of Mathieu Functions is discussed. The method followed is that of solving infinite equations.

Now we have.‡

$$ce_{2r}(\theta, q) = ce_{2r}\left(\frac{\pi}{2}, q\right) \sum_{n=0}^{\infty} (-)^n\, \phi_{2r}(n)\ \ J_{2n}(kh\cos\theta).$$

for all r.

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† Dongall: Loc. cit.
† Dhar—Loc. cit.
* Goldstein—Loc. cit.
‡ Dhar—Loc. cit., p. 55.

Hence solving the infinite equations for $r=0, 1, \ldots\ldots$

$$ce_0(\theta, q) - ce_0\left(\frac{\pi}{2}, q\right)[\phi_0(0)\, J_0(kh\cos\theta) - \phi_0(1)\, J_2(kh\cos\theta) \ldots\ldots + (-)^n \phi_0(n)\, J_{2n}(kh\cos\theta) \ldots\ldots\ldots\ldots]$$

$$ce_{2r}(\theta, q) - ce_{2r}\left(\frac{\pi}{2}, q\right)[\phi_{2r}(0)\, J_0(kh\cos\theta) - \ldots\ldots\ldots\ldots\ldots\ldots + (-)^n \phi_{2r}(n)\, J_{2n}(kh\cos\theta) + \ldots\ldots\ldots\ldots\ldots]$$

we have, writing J_{2n} for $J_{2n}(kh\cos\theta)$.

$$\frac{J_0}{D_0} = \frac{-J_1}{D_1} = \frac{J_2}{D_2} = \ldots = \frac{(-)^n J_{2n}}{D_n} = \ldots = \frac{-1}{\Delta}$$

where $D_0, D_1 \ldots\ldots$ and Δ are infinite determinate and

$$\Delta = \begin{vmatrix} ce_0\left(\frac{\pi}{2}, q\right)\phi_0(0), & ce_0\left(\frac{\pi}{2}, q\right)\phi_0(1), \ldots\ldots\ldots\ldots\ldots \\ ce_2\left(\frac{\pi}{2}, q\right)\phi_2(0), & ce_2\left(\frac{\pi}{2}, q\right)\phi_2(1), \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ ce_{2r}\left(\frac{\pi}{2}, q\right)\phi_{2r}(0), \ldots\ldots\ldots\ldots\ldots ce_{2r}\left(\frac{\pi}{2}, q\right)\phi_{2r}(r) \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix}$$

But $a_{0,r}\,\phi_{2r}(n) = a_{n,r}$.

Hence, $\Delta = \dfrac{ce_0\left(\frac{\pi}{2}, q\right) ce_2\left(\frac{\pi}{2}, q\right) \ldots\ldots\ldots ce_{2r}\left(\frac{\pi}{2}, q\right) \ldots\ldots}{a_{0,0}\, a_{1,1} \ldots\ldots\ldots\ldots\ldots\ldots a_{0,r} \ldots\ldots\ldots} \times$

$$\begin{vmatrix} a_{0,0}, & a_{1,0}, & a_{2,0} \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{0,1}, & a_{1,1}, & a_{2,1} \ldots\ldots & \ldots\ldots & & \\ a_{0,2}, & a_{1,2}, & a_{2,2} \ldots\ldots & \ldots\ldots & & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots a_{3,3} & \ldots\ldots & & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ a_{0,r}, & a_{1,r} & \ldots\ldots & \ldots\ldots & \ldots\ldots & a_{r,r} \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \ldots\ldots \end{vmatrix}$$

Now in the expansion of $ce_{2r}(\theta, q)$ the co-efficient of $\cos 2r\theta$ is unity. Hence

$$a_{r,r} = 1 \text{ for all values of } r.$$

Hence the product of all the diagonal elements is unity. The sum of the non-diagonal elements can be shown to be convergent. The sum of the elements of the r^{th} row except the term in the leading diagonal is equal to

$$\sum_{s=0}^{\infty} a_{s,\,r-1} = ce_{2r}\,(o,\,q)-1.$$

Now $ce_{2r}(o,\,q) \to 1$ as $q \to 0$.

Hence $|\,ce_{2r}(o,\,q)-1\,| < \frac{\epsilon}{2^{r+1}}$ when $q < \eta$.

And this is true for every other row.

Hence the sum of the non-diagonal elements is less than

$$\sum_{r=0}^{\infty} \frac{\epsilon}{2^{r+1}} = \epsilon. \text{ (a finite quantity).}$$

The absolute convergence of the series follows easily by a slight change in the above argument. The multiplying factor of the determinant is

$$\prod_{n=0}^{\infty} \left\{ \frac{ce_{2n}\left(\frac{\pi}{2},\,q\right)}{a_{o,\,n}} \right\}.$$

But for all values of n, $\frac{ce_{2n}\left(\frac{\pi}{2},\,q\right)}{a_{o,\,n}}$ is bounded. Hence we can choose a finite quantity A, so that

$$\left| \frac{ce_{2n}\left(\frac{\pi}{2},\,q\right)}{a_{o,\,n}} \right| < A^{\frac{1}{2^{n+1}}}$$ for all n and for q sufficiently small.

Hence $\prod_{n=0}^{\infty} \left\{ \frac{ce_{2n}\left(\frac{\pi}{2},\,q\right)}{a_{o,\,n}} \right\} < A^{\sum_{0}^{\infty} \frac{1}{2^{n+1}}} = A.$

Hence Δ is finite. ‖ Similarly it can be shown that each of $D_o, D_1, \ldots\ldots D_n, \ldots\ldots$ is convergent. The determinant D_n will contain in one of its columns the elements $ce_o(\theta,\,q), ce_2(\theta,\,q) \ldots$. $ce_{2r}(\theta,\,q) \ldots$. etc. If we expand D_n in terms of the elements of this column, we get

$$J_{2n}(kh\,\cos\theta) = (-)^{n+1} \frac{D_n}{\Delta}$$

‖ Whittaker and Watson—Modern Analysis, p. 2·8.

$$= \tfrac{1}{2} A_0\, ce_0(\theta, q) + A_1\, ce_2(\theta, q) + \ldots + A_r\, ce_{2n}(\theta, q) \ldots$$

where $A_0, A_1 \ldots$ etc. are coefficients which are functions of q and n.

VI. To bring out the dependence of the co-efficients of the expansion in the previous section we write the expansion as

$$J_{2n}(kh \cos\theta) = \tfrac{1}{2} A_{0,\,2n}\, ce_0(\theta, q) + \sum_{m=1}^{\infty} A_{m,\,2n}\, ce_{2m}(\theta, q). \quad (1)$$

To determine the coefficients $A_{m,\,2n}$, we multiply both sides by $ce_{2m}(\theta, q)$ and integrate term by term with respect to θ from 0 to 2π. Therefore,

$$\int_0^{2\pi} J_{2n}(kh \cos\theta).\ ce_{2m}(\theta, q) = \tfrac{1}{2} A_{0,\,2n} \int_0^{2\pi} ce_0(\theta, q) \times ce_{2m}(\theta, q)\, d\theta + \sum_{r=1}^{\infty} A_{r,\,2n} \int_0^{2\pi} ce_{2r}(\theta, q).\ ce_{2m}(\theta, q)\, d\theta.$$

Now the Mathieu Functions of Whittaker type, $ce_{2n}(\theta, q)$ are orthogonal*. Therefore,

$$\int_0^{2\pi} ce_{2m}(\theta, q).\ ce_{2n}(\theta, q)\, d\theta = 0 \text{ if } m \neq n.$$

and $\int_0^{2\pi} [ce_{2n}(\theta, q)]^2\, d\theta = \text{constant if } m = n.$

If following Goldstein† we adopt the convention that

$$\int_0^{2\pi} [ce_{2n}(\theta, q)]^2\, d\theta = \pi \quad n \neq 0.$$

and $\int_0^{2\pi} [ce_0(\theta, q)]^2\, d\theta = 2\pi \quad n = 0.$

we get, $\pi A_{m,\,2n} = \int_0^{2\pi} J_{2n}(kh \cos\theta).\ ce_{2m}(\theta, q)\, d\theta.$

But

$$ce_{2m}(\theta, q) = ce_{2m}\left(\frac{\pi}{2}, q\right) \sum_{r=0}^{\infty} (-)^r \phi_{2m}(r).\ J_{2r}(kh \cos\theta) \ldots (2)$$

Where $\phi_{2m}(r)$ has the same meaning as in section IV. But the series (2) is already proved to be uniformly convergent in section IV. Hence both sides of (2) be multiplied by a bounded function $J_{2m}(kh \cos\theta)$ and integrated term by term Therefore,

$$\pi A_{m,\,2n} = \int_0^{2\pi} ce_{2m}(\theta, q).\ J_{2n}(kh\cos\theta)\, d\theta.$$

$$= \int_0^{2\pi} ce_{2m}\left(\frac{\pi}{2}, q\right) \sum_{r=0}^{\infty} (-)^r \phi_{2m}(r) J_{2r}(kh\cos\theta) J_{2n}(kh\cos\theta)\, d\theta.$$

$$= \sum_{r=0}^{\infty} ce_{2m}\left(\frac{\pi}{2}, q\right) (-)^r \phi_{2m}(r) \int_0^{2\pi} J_{2r}(kh\cos\theta)\, J_{2n}(kh\cos\theta) d\theta.$$

$$= 2\, ce_{2m}\left(\frac{\pi}{2}, q\right) \sum_{r=0}^{\infty} (-)^r \phi_{2m}(r) B_{2r,\,2n} \cdots\cdots\cdots (3).$$

Hence

$$J_{2n}(kh\cos\theta) = \tfrac{1}{2} A_{0,\,2n}\, ce_0(\theta, q) + \sum_{m=1}^{\infty} A_{m,\,2n}\, ce_{2m}(\theta, q) \text{ where}$$

$$A_{m,\,2n} = \frac{2}{\pi} ce_{2m}\left(\frac{\pi}{2}, q\right). \sum_{r=0}^{\infty} (-)^r \phi_{2m}(r) \quad B_{2r,\,2n}$$

and $B_{2r,\,2m}$ is given by section III.

VII. We are now in a position to justify term by term the integration of the series (1) of the preceding section. It will be enough to prove the uniform convergence of the series where $A_{m,\,2n}$ is given by (3).

The sum of series (1) is less than the sum of

$$\sum_{m=0}^{\infty} |A_{m,\,2n}|\ |ce_{2m}(\theta, q)|.$$

Now all $ce_{2m}(\theta, q)$ are bounded for sufficiently small values of q.

Hence $|ce_{2m}(\theta, q)| < k$ for all m.

Hence the sum of the series (1) is less than that of

$$K \sum_{m=0}^{\infty} |A_{m,\,2n}| < \frac{2k}{\pi} \sum_{m=0}^{\infty} \left| \sum_{r=0}^{\infty} ce_{2m}\left(\frac{\pi}{2}, q\right) \phi_{2m}(r) \quad B_{2r,\,2n} \right|$$

$$< \frac{2k}{\pi} \sum_{m=0}^{\infty} \left| \sum_{r=0}^{\infty} \right| ce_{2m}\left(\frac{\pi}{2}, q\right) | \ |\phi_{2m}(r)|\ |B_{2r,\,2n}|.$$

To discuss the convergence of the double series† on the right hand side, it is sufficient to show that the limit of the sum of the terms in the square formed by the first m rows and m columns is finite. If $S_{m,\,m}$ denotes the required sum,

---

† Bromwich Infinite Series, Chap. V.

$$S_{m,\,m} = |\,ce_0\left(\tfrac{\pi}{2}, q\right)\phi_0(0)|\,|B_{0,\,2n}\,| + |\,ce_0\left(\tfrac{\pi}{2}, q\right)|\,|\,\phi_0(1)|\,|B_{2,\,2n}\,| + \ldots\ldots\ldots$$

$$+ |\,ce_2\left(\tfrac{\pi}{2}, q\right)\phi_2(0)|\,|B_{0,\,2n}\,| + |\,ce_2\left(\tfrac{\pi}{2}, q\right)|\,|\,\phi_2(1)|\,|B_{2,\,2n}\,| + \ldots\ldots\ldots$$

$$= |B_{0,\,2n}\,|\left[\,|ce_0\left(\tfrac{\pi}{2}, q\right)|\,\phi_0(0)\,| + |\,ce_2\left(\tfrac{\pi}{2}, q\right)|\,\phi_2(0)\,| \ldots\ldots\ldots m \text{ terms}\right]$$

$$+ |\,B_{2,\,2n}\,|\left[\,|\,ce_0\left(\tfrac{\pi}{2}, q\right)|\,|\,\phi_0(1) + |\,ce_2\left(\tfrac{\pi}{2}, q\right)|\,\phi_2(1)\,| + \ldots\ldots\ldots \;\;,,\;\;\right]$$

Each term within the brackets is less than K a finite quantity.

Hence $S_{m,\,m} < K \sum_{m=0}^{m} m\,|\,B_{2m,\,2n}\,|$

$$\therefore \operatorname*{Lt}_{m\to\infty} S_{m,\,m} < K \sum_{m=0}^{\infty} m\,|\,B_{2m,\,2n}\,|.$$

Now $\dfrac{u_m}{u_{m+1}}$ for the series on the right hand side

$$= \frac{m}{m+1}\cdot\frac{|\,B_{2m,\,2n}\,|}{|\,B_{2m+2,\,2n}\,|}$$

If we compare the corresponding terms of $|B_{2m,\,2n}\,|$ and $|\,B_{2m+2,\,2n}\,|$, it can be very easily seen that every term of $|B_{2m,\,2n}\,|$ is greater than the corresponding term of $|B_{2m+2,\,2n}\,|$ Hence

$$\frac{|\,B_{2m,\,2n}\,|}{|\,B_{2m+2,\,2n}\,|} > 1 \text{ for all } m.$$

$$\therefore \operatorname*{Lt}_{m\to\infty} \frac{u_m}{u_{m+1}} > 1.$$

Hence the series on the right hand side of $S_{m,\,m}$ is convergent and so $\operatorname*{Lt}_{m\to\infty} S_{m,m}$ is finite. Therefore, the series (1) is uniformly convergent.

VIII. The expansion investigated in the last section will now be used to obtain a property of Mathieu Functions.

Now $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = J_0(z) + \sum_{n=1}^{\infty} \left\{ t^n + (-)^n t^{-n} \right\} J_n(z),$

Put $t = \pm e^{i\theta}$ we get after simplification;

$$e^{\pm iz \sin\theta} = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) \cos 2n\theta \pm 2i \sum_{n=0}^{\infty} J_{2n+1}(z). \sin(2n+1)\theta.$$

Adding the two results and putting $\theta = 0$ we have

$$1 = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) \quad \ldots\ldots\ldots\ldots (1).$$

Substitute $kh \cos\theta$ for z. (1) reduces to

$$1 = J_0(kh\cos\theta) + 2\left\{ J_2(kh\cos\theta) + J_4(kh\cos\theta) + \ldots \right\} \ldots\ldots (2).$$

Expanding each of J_{2n} in terms of Mathieu functions by (1) and (3) of section V.

$$\begin{aligned} 1 = {} & \tfrac{1}{2} A_{00}\, ce_0(\theta, q) + A_{1,0}\, ce_2(\theta, q) + A_{2,0}\, ce_4(\theta, q) + \ldots\ldots \\ & + 2\,[\tfrac{1}{2} A_{0,2}\, ce_0(\theta, q) + A_{1,2}\, ce_2(\theta, q) + A_{2,2}\, ce_4(\theta, q) + \ldots\ldots \\ & \quad + \tfrac{1}{2} A_{0,4}\, ce_0(\theta, q) + A_{1,4}\, ce_2(\theta, q) + A_{2,4}\, ce_4(\theta, q) + \ldots \\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots] \ldots\ldots (3) \end{aligned}$$

If q be taken so small that its powers higher than 3 be neglected, we have,

$$\left.\begin{aligned} B_{0,0} &= \pi\left[1 + 2^3 q + 3^2 \cdot 2^2 \cdot q^2 + \frac{5^2 \cdot 2^5}{3^2} q^3 \ldots\right] \\ B_{0,2} &= B_{2,0} = \pi\left[-2q - 2^4 q^2 - 5oq^3 \ldots\right] \\ B_{2,2} &= \pi\left[6q^2 + \frac{5 \cdot 2^4}{3} q^3 \ldots\ldots\right] \\ B_{4,0} &= B_{0,4} = \pi\left[q^2 + 8q^3 \ldots\ldots\right] \\ B_{4,2} &= -\pi\,\frac{10}{3} q^3 \ldots\ldots\ldots] \\ B_{0,6} &= -\pi\,\frac{2}{9} q^3 \ldots\ldots, \end{aligned}\right\} \ldots\ldots (4)$$

Again if we take the expansions of $ce_0(\theta, q)$, $ce_2(\theta, q)$, $ce_4(\theta, q)$ as $ce_0(\theta, q) = 1 + [4q - 28q^3 + O(q^5)] \cos 2\theta + [2q^2 + O(q^4)] \cos 4\theta + \ldots\ldots\ldots$

$$ce_2(\theta, q) = [-2q + \frac{4\theta}{3} q^3 + O(q^5)] + \cos 2\theta + \left[\frac{2q}{3} + \right.$$

$$\frac{516}{3^2(0)^2} q^8 + O(q^8)] \cos 4\theta + \ldots\ldots\ldots$$

$$ce_4(\theta, q) = \left[\frac{1}{3} q^2 + O(q^4)\right] - \left[\frac{2q}{3} + \frac{14}{135} q^3 + \ldots\right] \cos 2\theta$$

$$+ \cos 4\theta + \left[\frac{2}{3} q + O(q^3)\right] \cos 6\theta + O(q^2) \cos 8\theta + \ldots\ldots$$

for which the values of a in the Mathieu's equation

$$\frac{d^2u}{d\theta^2} + (a + 16q \cos 2\theta)\, u = 0.$$

are (1) $-32q^3 + 224q^4 + O(q^6)$.

(2) $4 + \frac{80}{3} q^2 - \frac{6104}{27} q^4 + O(q^6)$

(3) $16 + \frac{32}{15} q^2 + O(q^4)$. ,

we get

$$\left.\begin{aligned}
&a_{0,0}\ \phi_0(0) = 1\\
&a_{0,0}\ \phi_0(1) = 4q - 28q^3 + \ldots\ldots\\
&a_{0,0}\ \phi_0(2) = 2q^2 \ldots\ldots\\
&a_{0,1}\ \phi_2(0) = -2q + \frac{40}{3} q^3 \ldots\ldots\ldots\\
&a_{0,1}\ \phi_2(1) = 1\\
&a_{0,1}\ \phi_2(2) = -\frac{2q}{3} + \ldots\ldots\\
&a_{0,2}\ \phi_4(0) = \frac{1}{3} q^2 \ldots\ldots\\
&a_{0,2}\ \phi_4(1) = -\frac{2}{3} q - \frac{14}{135} q^3 \ldots\ldots\\
&a_{0,2}\ \phi_4(2) = 1\\
&\qquad\qquad \text{etc.}
\end{aligned}\right\} \ldots\ldots (5).$$

Using (4) and (5) we find the values of $A_{n,2m}$ for different values of n and m.

$$\left.\begin{aligned}
A_{0,0} &= 2\, ce_0\left(\tfrac{\pi}{2}, q\right) \left[1 + 8q \times 44q^2 + \frac{2^5 \cdot 43}{3^2} q^3 \ldots\right]\\
A_{0,2} &= ce_0\left(\tfrac{\pi}{2}, q\right) \left[-4q - 2^5 q^2 - 2^2 \cdot 37 q^3 \ldots\ldots\right]\\
A_{0,4} &= ce_0\left(\tfrac{\pi}{2}, q\right) \left[2q^2 + 2^4 q^3 \ldots\ldots\ldots\ldots\right]\\
A_{0,6} &= ce_0\left(\tfrac{\pi}{2}, q\right) \left[-\frac{2^2}{3^2} q^3 \ldots\ldots\ldots\ldots\right]
\end{aligned}\right|$$

$$A_{1,0}=\frac{ce_2\left(\frac{\pi}{2},q\right)}{a_{0,1}}\ [-2^4q^3\ldots\ldots\ldots\ldots\ldots\ldots]$$

$$A_{1,2}=\frac{ce_2\left(\frac{\pi}{2},q\right)}{a_{0,1}}\ [-2^2q^2\times\frac{2^5}{3}q^3\ldots\ldots\ldots\ldots\ldots]$$

$$A_{1,4}=\frac{ce_2\left(\frac{\pi}{2},q\right)}{a_{0,1}}\ \left[\frac{2^3}{3}q^3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$A_{2,0}=\frac{ce_4\left(\frac{\pi}{2},q\right)}{a_{0,2}}\ [O(q^4)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots]$$

$$A_{2,2}=\frac{ce_4\left(\frac{\pi}{2},q\right)}{a_{0,2}}\ O(q^4)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots]\qquad\ldots(6)$$

Rewriting (3) in a different form, we have

$$\begin{aligned}1=&\ ce_0(\theta,q)\ [\tfrac{1}{2}A_{0,0}+A_{0,2}+A_{0,4}+A_{0,6}+\ldots]\\&+ce_2(\theta,q)\ [A_{1,0}+2A_{1,2}+2A_{1,4}\ \ldots\ldots\ldots]\\&+ce_4(\theta,q)\ [A_{2,0}+2A_{2,2}\ \ldots\ldots\ldots\ldots\ldots]\ldots(7)\end{aligned}$$

Now it is easy to show all the terms in the square brackets except those that occur in (6) begin with powers higher than 3, and hence neglecting them we get

$$\text{coefficient of } ce_0(\theta,q)=ce_0\left(\frac{\pi}{2},q\right)\ [1+4q+14q^2\times\frac{184}{9}q^3]$$

$$\text{coefficient of } ce_2(\theta,q)=\frac{ce_2\left(\frac{\pi}{2},q\right)}{a_{0,1}}\left[-8q^2+\frac{2^5q^3}{3}\right]$$

other coefficients are to be neglected as they contain q^4 etc. Hence (7) reduces to.

$$\begin{aligned}1=&\ [1+4q+14q^2+\frac{184}{9}q^3]\ ce_0\left(\frac{\pi}{2},q\right)\ ce_0(\theta,q)\\&+[-8q^2+\frac{32}{3}q^3]\frac{ce_2\left(\frac{\pi}{2},q\right)}{a_{0,1}}\ ce_2(\theta,q)\ \ldots\ldots\ldots\ (8)\end{aligned}$$

a relation between $ce_0(\theta,q)$ and $ce_2(\theta,q)$ when q is sufficiently small.

Equation (8) can be put in a more elegent form, if the values of $a_{0,1}$, $ce_0\left(\frac{\pi}{2},q\right)$ and $ce_2\left(\frac{\pi}{2},q\right)$ are substituted and powers of q higher than 3 are neglected we will then get.

$$1=ce_0(\theta,q)-[4q+\frac{16}{3}q^2+22q^3]\ \ ce_2(\theta,q)\ldots\ldots\ (9)$$

It is interesting to note that the limit of the right hand side is unity as q tends to zero.

It is possible to establish relations between ce_0, ce_2, ce_4 .... etc. by this method, but the calculations are very tedious. Similar and other relations and expansions of other types are under investigation and I hope to communicate the results soon. In conclusion, I must thank Dr. Dhar for his valuable suggestions.

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